TESTING FOR CORRELATION IN ERROR-COMPONENT MODELS

SUPPLEMENT

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Power calculations for a dynamic regression model. Here we examine a stationary first-order autoregression and estimate the slope coefficient by the instrumental-variable estimator of Anderson and Hsiao (1981). This is the optimal generalized method-of-moment estimator constructed from the moments

$$\mathbb{E}\begin{pmatrix} y_{g,0}\Delta\eta_{g,2} \\ y_{g,1}\Delta\eta_{g,3} \end{pmatrix} = \mathbf{0}.$$

In the calculations to follow we assume that $\alpha_g = 0$ for all groups. The Jacobian and covariance matrix of the two Anderson and Hsiao (1981) moments are, respectively, equal to

$$\boldsymbol{G} := \frac{\sigma^2}{1+\beta}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \boldsymbol{A} := \frac{\sigma^4}{1-\beta^2}\begin{pmatrix} 2 & -\beta \\ -\beta & 2 \end{pmatrix},$$

when the errors are uncorrelated and have variance σ^2. These matrices can be combined to find that

$$\omega_g = -(\boldsymbol{G}'\boldsymbol{A}^{-1}\boldsymbol{G})^{-1}\boldsymbol{G}'\boldsymbol{A}^{-1}\begin{pmatrix} y_{g,0}\Delta\eta_{g,2} \\ y_{g,1}\Delta\eta_{g,3} \end{pmatrix} = -\frac{1+\beta}{\sigma^2}\frac{y_{g,0}\Delta\eta_{g,2} + y_{g,1}\Delta\eta_{g,3}}{2}.$$

When the moment conditions hold this is a mean-zero random variable. The moments typically fail to hold when our null of no within-group correlation is violated. When

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$\varepsilon_{g,i} = u_{g,i} + \theta\, u_{g,i-1}$ with $u_{g,i} \sim$ independent $(0, \sigma^2)$,

$$\mathbb{E}\begin{pmatrix} y_{g,0} \Delta\eta_{g,2} \\ y_{g,1} \Delta\eta_{g,3} \end{pmatrix} = -\theta\, \sigma^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

so that $\mathbb{E}(\omega_g) = (1+\beta)\,\theta$. This is indeed non-zero for all $\theta \neq 0$. When the errors, instead, follow the autoregressive process $\varepsilon_{g,i} = \rho\, \varepsilon_{g,i-1} + u_{g,i}$ with $u_{g,i} \sim$ independent $(0, \sigma^2)$, the bias equals

$$\mathbb{E}\begin{pmatrix} y_{g,0} \Delta\eta_{g,2} \\ y_{g,1} \Delta\eta_{g,3} \end{pmatrix} = -\frac{\varrho}{1-\rho\beta}\, \sigma^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix};$$

this can be verified using that $y_{g,i} = \sum_{j=0}^{\infty} \beta^j \varepsilon_{g,i-j}$, which itself follows from backward substitution. Hence, $\mathbb{E}(\omega_g) = (1+\beta)\,\varrho/(1-\rho\beta)$ in this case. This again fails to be zero whenever ρ is non-zero.

The Jacobian of $\mathbb{E}(\boldsymbol{v}_g)$ with respect to β is

$$\boldsymbol{\Omega} = (1-\beta)\,\sigma^2 \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

which has a simple form. It follows that, when errors follow a moving-average process and an autoregressive process, respectively,

$$\mathbb{E}(\boldsymbol{v}_g + \boldsymbol{\Omega}\,\omega_g) = \theta\,\sigma^2\,\beta^2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad \mathbb{E}(\boldsymbol{v}_g + \boldsymbol{\Omega}\,\omega_g) = \frac{\beta-\rho}{1-\beta\rho}\,\beta\,\varrho\,\sigma^2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The bias expressions show that our test will have no power when $\beta = 0$. When the errors follow a first-order autoregressive process trivial power will also occur when $\rho = \beta$. The bias in the Anderson and Hsiao (1981) estimator effectively cancels the bias in our moment conditions in these cases. This result is not general, in that it is specific to the setting of a three-wave panel, stationary data, and the use of the Anderson and Hsiao (1981) estimator.

Lengthy but standard calculations show that the covariance matrix of $\boldsymbol{v}_g + \boldsymbol{\Omega}\,\omega_g$ under the null is

$$\tilde{\boldsymbol{V}} = \sigma^4 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} - \frac{1-\beta^2}{2}\sigma^4 \begin{pmatrix} 2+\beta & -(1+\beta) \\ -(1+\beta) & \beta \end{pmatrix}.$$

Figure 1: Power calculations using the Anderson-Hsiao estimator

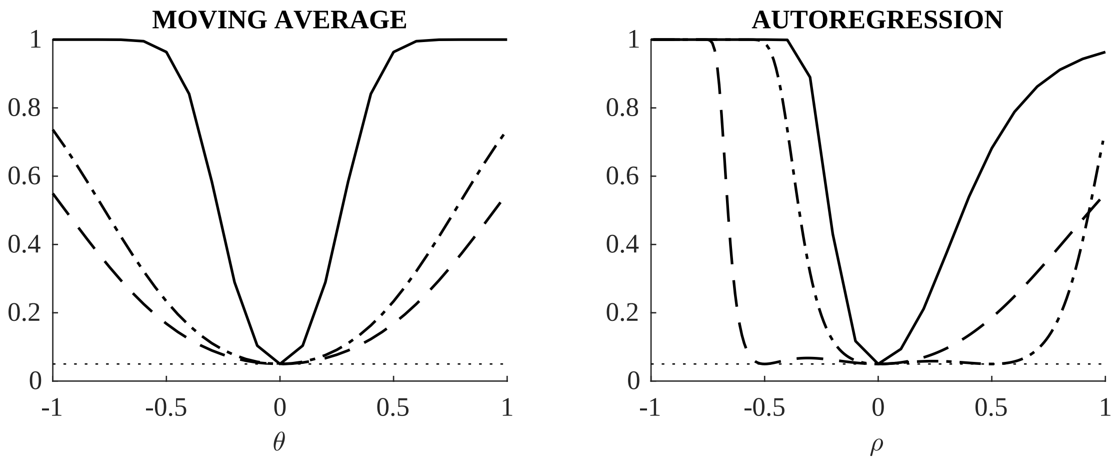

Figure notes. Our test for $\beta = -\frac{1}{2}$ (dashed line) and $\beta = \frac{1}{2}$ (dashed-dotted line) using the Anderson-Hsiao estimator, together with its oracle version (solid line). The size of the tests (.05) is indicated by a horizontal dotted line.

The first matrix in the right-hand side expression corresponds to $\boldsymbol{V}$ from above (by virtue of stationarity). With $|\beta| < 1$ it follows that $(\tilde{\boldsymbol{V}})_{11} < (\boldsymbol{V})_{11}$ and $(\tilde{\boldsymbol{V}})_{12} < (\boldsymbol{V})_{12}$ (in magnitude) for any β. On the other hand, $(\tilde{\boldsymbol{V}})_{22} < (\boldsymbol{V})_{22}$ when $\beta > 0$ and $(\tilde{\boldsymbol{V}})_{22} > (\boldsymbol{V})_{22}$ when $\beta < 0$.

The non-centrality parameter in Theorem 2(ii) is then found to equal

$$\lambda_\beta \, \beta^2 \, \theta^2, \qquad \lambda_\beta \, \varrho^2 \left(\frac{\beta - \rho}{1 - \beta\rho} \right)^2,$$

for moving-average and autoregressive alternatives, respectively, where we have used the shorthand

$$\lambda_\beta := \frac{4\beta^2}{2\beta^3 + 3\beta^2 - 2\beta + 3}$$

We note that λ_β is roughly U-shaped on $(-1, 1)$, reaching its minimum of zero at zero and its maximum of $2/3$ at the boundary. This implies, for example, that the test is uniformly less powerful against moving-average alternatives compared to the case where the errors are directly observed.

We illustrate our power calculations for this example in Figure 1, again for $n = 100$. The power curves are for $\beta = -\frac{1}{2}$ (dashed line) and $\beta = \frac{1}{2}$ (dashed-dotted line). The solid line corresponds to the power curve for the oracle test where β is known (and so Theorem 1 applies); these curves co-incide with those reported in Figure **??**. The plots illustrate the power loss relative to the oracle and the dependence of power on the value of the autoregressive coefficient. In the autoregressive case it also shows the loss of power against alternatives where ρ is close to β.

Under the null the Anderson and Hsiao (1981) estimator is inefficient as we equally have

$$\mathbb{E}(y_{g,0}\Delta\eta_{g,3}) = 0$$

in that case. Combining both sets of moments would yield the Arellano and Bond (1991) estimator. To show the sensitivity of our test to the first-step estimator used we next evaluate local power when using the (just identified) estimator based on this additional moment condition alone. It turns out that the rank condition for this estimator fails when $\beta = 0$ and so, in what follows, we presume that $\beta \neq 0$. The intermediate calculations are similar to before and omitted for brevity. We have

$$\tilde{\boldsymbol{V}} = \sigma^4 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} - \frac{\sigma^4}{2}\frac{1-\beta^2}{\beta^2} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

For moving-average alternatives we find that

$$\mathbb{E}(\boldsymbol{v}_g + \boldsymbol{\Omega}\boldsymbol{\omega}_g) = \mathbb{E}(\boldsymbol{v}_g) = \theta\,\sigma^2 \begin{pmatrix} -1 \\ 1 \end{pmatrix};$$

indeed, $\mathbb{E}(y_{g,0}\Delta\eta_{g,3}) = 0$ remains valid under such alternatives, and so $\mathbb{E}(\boldsymbol{\omega}_g) = 0$ here. For autoregressive alternatives on the other hand,

$$\mathbb{E}(\boldsymbol{v}_g + \boldsymbol{\Omega}\boldsymbol{\omega}_g) = \varrho\,\sigma^2\,\frac{2\beta - \rho\beta^2 - \rho}{2\beta(1-\beta\rho)} \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

which is again more complicated. Here, our test will have no power when $\rho = 2\beta/(1+\beta^2)$.

The plots in Figure 2 compare the power curves of this alternative implementation of our test to the former as well as its oracle version. The power gains are substantial in the case of moving-average alternatives. There is a small power loss relative to the oracle as the

Figure 2: Power calculations using the alternative estimator

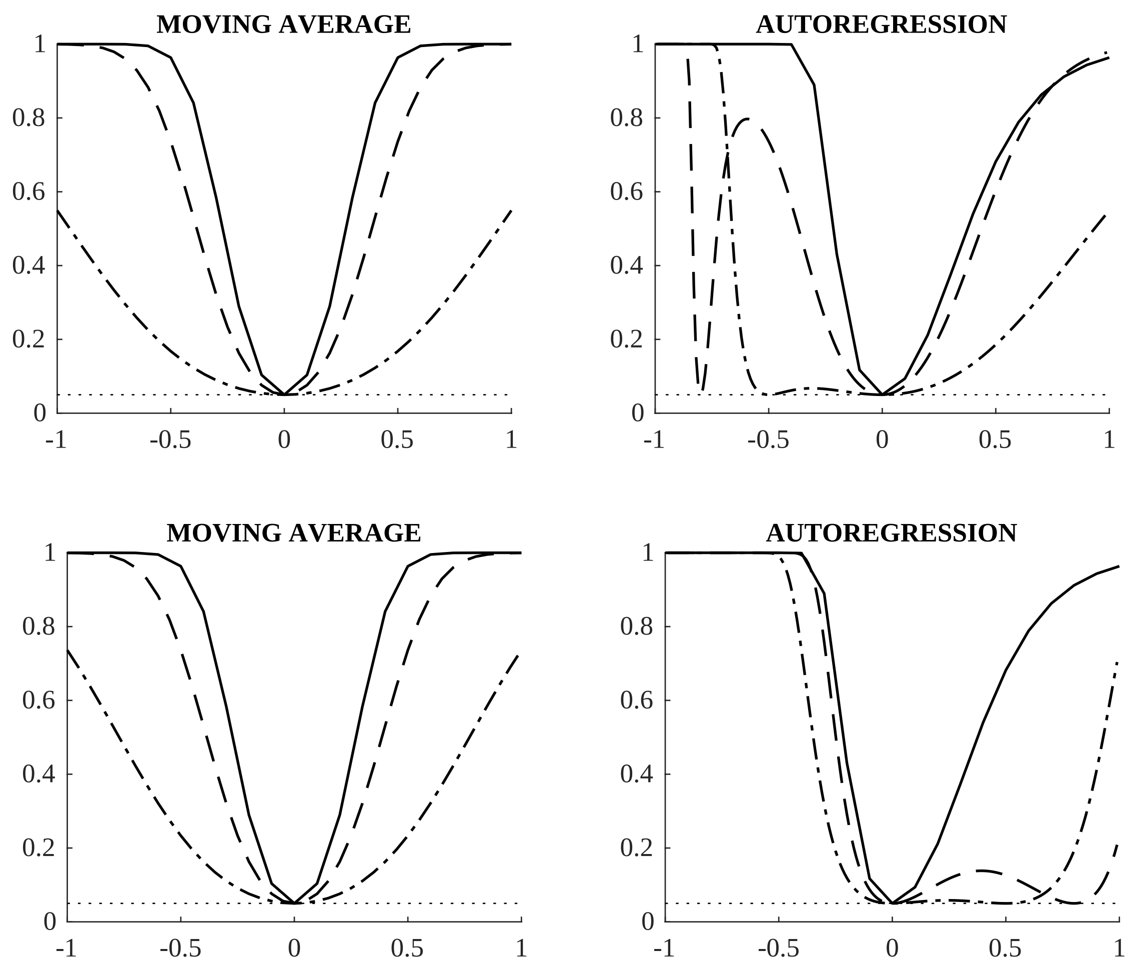

Figure notes. Our test (dashed line) for $\beta = -\frac{1}{2}$ (upper two plots) and $\beta = \frac{1}{2}$ (lower two plots) using the alternative method-of-moment estimator, together with its oracle version (solid line) and the test using the Anderson-Hsiao estimator (dashed-dotted line). The size of the tests (.05) is indicated by a horizontal dotted line.

estimator of β introduces additional sampling noise. The variance increase depends on β only through its square and so the power curves for $\beta = -\frac{1}{2}$ in the upper plot and for $\beta = \frac{1}{2}$ in the lower plot co-incide. The relative power gains against autoregressive alternatives are a more complicated function of the parameter values but are still present over most parts of the $(-1, 1)$ interval.

References

Anderson, T. W. and C. Hsiao (1981). Estimation of dynamic models with error components. *Journal of the American Statistical Association 76*, 598–606.

Arellano, M. and S. Bond (1991). Some tests of specification for panel data: Monte Carlo evidence and an application to employment equations. *Review of Economic Studies 58*, 277–297.